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LETTER TO THE EDITOR

Topological properties of linked disclinations in anisotropic liquids*

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Abstract. A connection between the topological properties of the O(3)- and SO(3)-σ models based on Hopf's invariant and the Wess–Zumino term, respectively, and related topological concepts are indicated. This is applied to the $O(3)/\mathbb{Z}_2$- and the $SO(3)/P_i$-σ models, where $\mathbb{Z}_2$ and $P_i \subset SO(3)$ are point symmetry groups describing anisotropic liquids. It is shown that the Hopf invariant for the nematic liquid assumes integer multiples of $\frac{1}{4}$ and similar results hold for the other σ-models. Applications to a topological field theory of $O(3)/\mathbb{Z}_2$- and $SO(3)/P_i$-σ models are indicated.

The order parameter of the O(3)-σ model assumes values on the 2-sphere S^2 and its defect configurations are usually classified by the three homotopy groups (see e.g. Mermin 1979 and Kléman 1983)

$$\pi_1(S^2) \simeq \mathrm{I} \qquad \pi_2(S^2) \simeq \mathbb{Z} \qquad \pi_3(S^2) \simeq \mathbb{Z} \tag{1}$$

where $\mathbb{Z}$ is the cyclic group of $\mathrm{ord}(\mathbb{Z}) = \infty$.

A 'measure' of the classes of $\pi_3(S^2)$ is given by the Hopf invariant (see e.g. Kundu and Rybakov 1982)

$$Q(\boldsymbol{n}) = -\frac{1}{(8\pi)^2} \int_M \mathrm{d}^3x\, \boldsymbol{A} \cdot \boldsymbol{B} \tag{2}$$

where $\{\boldsymbol{n}\}$ is a unit vector field, and $\boldsymbol{A}$ and $\boldsymbol{B}$ are suitable defined vector potential and magnetic induction, respectively of an U(1)-gauge theory; M is taken initially as a closed and simply connected 3-space, i.e. $\pi_1(M) \simeq \mathrm{I}$. Q can be identified with the Gauss linking number of disclination loops, which are the line defects of the system (the latter are defined by the border lines of 'cut surfaces' fixing the configuration of $\{n\}$ (Kundu and Rybakov 1982, Holz 1991)). Due to $\pi_1(S^2) \simeq \mathrm{I}$ disclinations are topologically unstable if they are unlinked, whereas in a knotted and linked configuration they are stable due to $\pi_3(S^2) \simeq \mathbb{Z}$. The 'time change' of (2) is defined by

$$\frac{\mathrm{d}Q}{\mathrm{d}t} \equiv \frac{2}{(8\pi)^2} \int_M \mathrm{d}^3x\, \boldsymbol{E} \cdot \boldsymbol{B} \tag{3}$$

where $\boldsymbol{E}$ is the 'electric' field strength. During topology changing processes ($\boldsymbol{E} \cdot \boldsymbol{B} \neq 0$) disclination lines intersect and 'magnetic' N-poles are generated due to $\pi_2(S^2) \simeq \mathbb{Z}$.

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During such processes A is a non-trivial U(1)-connection and Q is not well defined; however, this does not apply to the instanton number η from which (3) is obtained. For an introduction to some topological concepts of relevance in this work, see Eguchi *et al* (1980) and Madore (1981).

Some of the properties indicated above will be worked out (borrowing ideas developed by Witten (1989), and Dijkgraaf and Witten (1990)) and extended to $O(3)/\mathbb{Z}_2$- and $SO(3)/P_i$-σ models, where $\mathbb{Z}_2$ and $\{P_i\}$ are binary- and point-symmetry groups of anisotropic liquids.

The order parameter $(O(3)/\mathbb{Z}_2)$ of the uniaxial nematic liquid assumes values in the projective 2-sphere P^2, where $\pi_1(P^2)\simeq\mathbb{Z}_2$ and $\pi_{2(3)}(P^2)\simeq\mathbb{Z}$ (Whitehead 1978). It displays accordingly also disclinations of strength $s\in\mathbb{Z}+\frac{1}{2}$, and which contain core singularities, requiring a modification of (2) and (3). Similar considerations apply to $SO(3)\simeq P^3$ (projective 3-sphere) and for which $\pi_1(P^3)\simeq\mathbb{Z}_2$, $\pi_2(P^3)\simeq \mathrm{I}$ and $\pi_3(P^3)\simeq\mathbb{Z}$ (Whitehead 1978). A 'measure' of the classes of $\pi_3(P^3)$ is given by the Wess–Zumino term

$$\Gamma_{\mathrm{WZ}}=\frac{1}{48\pi^2}\int_M \mathrm{d}^3x\,\varepsilon^{pqr}\,\mathrm{trace}\{(\mathbf{R}^t\partial_p\mathbf{R})(\mathbf{R}^t\partial_q\mathbf{R})(\mathbf{R}^t\partial_r\mathbf{R})\} \tag{4}$$

where ε^{pqr} is the totally antisymmetric symbol $p=1,2,3$, etc; $\mathbf{R}(\{\boldsymbol{n}^a\}_{a=1,2,3})\in SO(3)$ is a 3×3 matrix and $\{\boldsymbol{n}^a\}_{a=1,2,3}$ an othonormal drei-bein field. Equation (4) can be reduced to (2), applying to each constituent $\boldsymbol{n}^a$ and with $Q(\boldsymbol{n}^a)\in\frac{1}{2}\mathbb{Z}$. These results are extended to the $SO(3)$-σ models.

In the following section the topological relation between the O(3)- and SO(3)-σ models is worked out, based on the Chern–Simons action (Eguchi *et al* 1980).

$$\Gamma_{\mathrm{CS}}=\frac{1}{32\pi^2}\int_M \mathrm{d}^3x\,\varepsilon^{pqr}\,\mathrm{trace}\{\mathbf{F}_{pq}\mathbf{A}_r-\tfrac{2}{3}\mathbf{A}_p\mathbf{A}_q\mathbf{A}_r\} \tag{5}$$

where

$$\mathbf{F}_{\mu\nu}=\partial_\mu\mathbf{A}_\nu-\partial_\nu\mathbf{A}_\mu+[\mathbf{A}_\mu,\mathbf{A}_\nu]=F^a_{\mu\nu}\mathbf{T}^a \qquad \mu=0,1,2,3.$$

Here $\{\mathbf{T}^a\}$ are the generators of the Lie-algebra of SO(3) in the 3×3 representation; $[\mathbf{T}^a,\mathbf{T}^b]=\varepsilon^{abc}\mathbf{T}^c$, where $\mathrm{trace}(\mathbf{T}^{a^2})=-2$, $a=1,2,3$, $\mathrm{trace}(\mathbf{T}^1\mathbf{T}^2\mathbf{T}^3)=-1$, and trace $(\mathbf{T}^a\mathbf{T}^b)=0$ for $a\neq b$. Γ_{CS} will be studied for the simple 'λ-connections':

$$\mathbf{A}_\mu(\lambda)=\lambda(\partial_\mu\mathbf{R}^t)\cdot\mathbf{R} \tag{6a}$$

where λ is some constant, and $\lambda=-1$ is the flat connection, which for $\pi_1(M)\simeq\mathrm{I}$ is unique in the sense that it is gauge equivalent modulo gauge transformations $G:\mathbf{A}\to\mathbf{G}(\mathbf{A})=\mathbf{G}A\mathbf{G}^t+\mathbf{G}\,\mathrm{d}\mathbf{G}^t$ to the trivial connection $\mathbf{A}\equiv0$. Here $\mathbf{G}\in\mathcal{G}$, and $\mathcal{G}$ is the group of continuous gauge transformations, i..e the maps $M\to SO(3)$, with $\pi_O(\mathcal{G})=\mathbb{Z}$. One easily obtains

$$\mathbf{F}_{\mu\nu}(\lambda)=(1+\lambda)(\partial_\mu\mathbf{A}_\nu-\partial_\nu\mathbf{A}_\mu) \tag{6b}$$

and

$$\Gamma_{\mathrm{CS}}(\lambda)=\frac{1}{16\pi^2}\left(1+\frac{2\lambda}{3}\right)\int_M \mathrm{d}^3x\,\varepsilon^{pqr}\,\mathrm{trace}((\partial_p\mathbf{A}_q)\mathbf{A}_r) \tag{7}$$

using the identities

$$\lambda(\partial_\mu A^a_\nu(\lambda)-\partial_\nu A^a_\mu(\lambda))=2\varepsilon^{abc}A^b_\mu(\lambda)A^c_\nu(\lambda) \tag{8}$$

which are independent of λ, for $\lambda \neq 0$. Equations (6a), (7), (8) are easily extended to the case where λ is replaced by a triple of numbers $\{\lambda^a\}$. Use of $\lambda = \lambda(x)$ implies that in (7) and all consecutive formulae λ-dependent factors have to be moved under the integral symbol; more involved formulae are obtained for

$$\mathbf{A}(\lambda(x)) = \sum_{a=1}^{3} \lambda^a(x) A^a(1) \mathbf{T}^a$$

where also $\partial_\mu \lambda^a$-terms do not cancel.

Setting

$$A_p^{a,\mathscr{H}} \equiv +2A_p^a(1) \qquad B_p^{a,\mathscr{H}} \equiv \varepsilon_{pqr} \partial_q A_r^{a,\mathscr{H}} \tag{9}$$

one obtains

$$\Gamma_{\mathrm{CS}}(\lambda) = 2\left(1 + \frac{2\lambda}{3}\right) \lambda^2 \sum_{a=1}^{3} Q^a \tag{7$'$}$$

where $Q^a \equiv Q(\boldsymbol{n}^a)$ is obtained from (2) by the replacement $(\boldsymbol{A}, \boldsymbol{B}) \to (\boldsymbol{A}^{a,\mathscr{H}}, \boldsymbol{B}^{a,\mathscr{H}})$. On the other hand, setting $(\mathrm{d}\mathbf{R}' \cdot \mathbf{R})_{ab} = \varepsilon_{abc} \Theta^c$, where $\Theta^c = \Theta_\mu^c \, \mathrm{d}x^\mu$, $c = 1, 2, 3$, are 1-forms, one obtains $A_\mu^a(-1) = -\Theta_\mu^a$ and

$$\Gamma_{\mathrm{CS}}(-1) = \Gamma_{\mathrm{WZ}} = \frac{1}{8\pi^2} \int \Theta^1 \Theta^2 \Theta^3 = \mathscr{N}. \tag{10}$$

Here $\mathscr{N} \in \mathbb{Z}$ is the winding number of $\mathbf{A}(-1)$ 'measuring' the classes of $\pi_3(\mathrm{SO}(3))$. Similarly one obtains

$$\Gamma_{\mathrm{CS}}(\lambda) = 3\left(1 + \frac{2\lambda}{3}\right) \lambda^2 \mathscr{N}. \tag{11}$$

Equation (11) implies that $\lambda = \frac{1}{2}, \pm 1, 3\mathbb{Z}$ yield $\Gamma_{\mathrm{CS}}(\lambda) \in \mathbb{Z}$, for all $\mathscr{N} \in \mathbb{Z}$; $\Gamma_{\mathrm{CS}}(0) = \Gamma_{\mathrm{CS}}(-\frac{3}{2}) = 0$, whereas for rational values of λ one has $\Gamma_{\mathrm{CS}}(\lambda) = (q(\lambda)/p(\lambda))\mathscr{N}$ and needs $\mathscr{N} \in p(\lambda)\mathbb{Z}$ for $q(\lambda)$ and $p(\lambda)$ relatively prime. For a gauge transformation G of $\mathbf{A}(\lambda)$, (11) changes by an integer ($\deg(G)$) independent of λ. Within this scheme a classification of topological SO(3)-σ models based on λ-connections is possible.

An alternative form of $\Gamma_{\mathrm{CS}}(\lambda)$ is obtained using the representation $\mathbf{R}(\{\boldsymbol{n}^a\}_{a=1,2,3})$. In that case $\mathbf{A}(1) = -\frac{1}{2}\varepsilon_{abc}\, \mathrm{d}\boldsymbol{n}^a \cdot \boldsymbol{n}^b \mathbf{T}^c$ yields ($\boldsymbol{n}^b_{,\mu} \equiv \partial_\mu \boldsymbol{n}^b$):

$$A_\mu^a(1) = -\tfrac{1}{2}\varepsilon_{abc} \boldsymbol{n}^b_{,\mu} \cdot \boldsymbol{n}^c \tag{12a}$$

$$F_{\mu\nu}^a(1) = 2\varepsilon_{abc} \boldsymbol{n}^b_{,\mu} \cdot \boldsymbol{n}^c_{,\nu}. \tag{12b}$$

It is now a simple matter to derive the identity

$$F_{\mu\nu}^a(1) = 2(\partial_\mu \boldsymbol{n}^a \times \partial_\nu \boldsymbol{n}^a) \cdot \boldsymbol{n}^a \tag{13}$$

inserting $\boldsymbol{n}^a = \varepsilon^{abc} \boldsymbol{n}^b \times \boldsymbol{n}^c$ into (12b). Comparison of (13) with the 'electromagnetic' representation of the O(3)-σ model (Kundu and Rybakov 1982) shows that

$$F_{\mu\nu}^a(1) = F_{\mu\nu}(\boldsymbol{n}^a) \equiv F_{\mu\nu}^{a,\mathscr{H}} \qquad 2A_\mu^a(1) = A_\mu(\boldsymbol{n}^a) \equiv A_\mu^{a,\mathscr{H}} \tag{14}$$

being consistent with (9).

Next the individual terms in (7$'$) are evaluated. This yields

$$\boldsymbol{A}^{a,\mathscr{H}} \cdot \boldsymbol{B}^{a,\mathscr{H}} = 4\varepsilon^{pqr} (\boldsymbol{n}^a_{,p} \cdot \boldsymbol{n}^b)(\boldsymbol{n}^b_{,q} \cdot \boldsymbol{n}^c)(\boldsymbol{n}^c_{,r} \cdot \boldsymbol{n}^a)$$

where (a, b, c) form a cyclic arrangement of $(1, 2, 3)$. Under the same condition $\Theta^a = -\mathrm{d}\boldsymbol{n}^b \cdot \boldsymbol{n}^c$ implies $\boldsymbol{A}^{a,\mathcal{H}} \cdot \boldsymbol{B}^{a,\mathcal{H}} = -4\Theta^a \wedge \Theta^b \wedge \Theta^c$. From this follows that each constituent of the drei-bein field $\{\boldsymbol{n}^a\}$ gives the same contribution to Γ_{CS}, yielding for a test

$$\Gamma_{\mathrm{CS}}(\lambda) = 2\left(1 + \frac{2\lambda}{3}\right)\lambda^2 \frac{1}{(8\pi)^2} 4 \times 3 \times \mathcal{N} \operatorname{vol}(\mathrm{SO}(3)) \tag{10'}$$

where $\operatorname{vol}(\mathrm{SO}(3)) = 8\pi^2$, and which is identical to (11). From (7') and (11) follows

$$Q^a \equiv -\frac{1}{(8\pi)^2} \int_M \mathrm{d}^3x\, \boldsymbol{A}^{a,\mathcal{H}} \cdot \boldsymbol{B}^{a,\mathcal{H}} = \mathcal{N}/2 \tag{15}$$

and implies that $\mathcal{N}$ must be divisible by 2 in order that the drei-bein field $\{\boldsymbol{n}^a\}_{a=1,2,3}$ is smooth. Naturally, this is related to $\pi_1(\mathrm{SO}(3)) \simeq \mathbb{Z}_2$, and that the respective disclinations display core singularities.

The present result is consistent with the property that a smooth SO(3)-bundle may trivially extend to a SU(2)-bundle. Due to $\mathrm{SU}(2) \simeq S^3$, and $P^3 \simeq S^3/\mathbb{Z}_2$ we have $\operatorname{vol}(\mathrm{SU}(2)) = 2\operatorname{vol}(\mathrm{SO}(3))$, and in a normalization where arbitrary integers $\kappa \in \mathbb{Z}$ are allowed for $\Gamma_{\mathrm{CS}}^{S^3}$ we have $\Gamma_{\mathrm{CS}}^{\mathrm{SO}(3)} = \Gamma_{\mathrm{CS}}^{\mathrm{SU}(2)}/2 = \kappa/2$:

$$Q^a_{\mathrm{SO}(3)} = \kappa/4. \tag{16}$$

Here the SO(3)-index is a reminder that the normalization has been changed. This implies that only those SO(3)-bundles, where κ is divisible by four extend trivially to SU(2)-bundles, and agrees with the result derived by Dijkgraaf and Witten (1990).

In the presence of singularities M cannot be considered anymore as a closed and simply connected 3-space and therefore $\Gamma_{\mathrm{CS}}(\lambda)$ has to be supplemented by an additional term. From $A^a_\mu(1)$ one obtains for the additional field strength tensor on the cylinder $M \times R$

$$\delta F^a_{\mu\nu}(1) = \tfrac{1}{2}\varepsilon_{abc}(\boldsymbol{n}^b{}_{,\mu\nu} - \boldsymbol{n}^b{}_{,\nu\mu}) \cdot \boldsymbol{n}^c \qquad (\nu, \mu) = 0, 1, 2, 3$$

implying

$$\delta E^a_i(1) = -\tfrac{1}{2}\varepsilon_{abc}(\boldsymbol{n}^b{}_{,0i} \cdot \boldsymbol{n}^c - \boldsymbol{n}^b{}_{,i0} \cdot \boldsymbol{n}^c)$$

$$\delta B^a_i(1) = \varepsilon^{ipq}\varepsilon_{abc}(\boldsymbol{n}^b{}_{,pq} - \boldsymbol{n}^b{}_{,qp}) \cdot \boldsymbol{n}^c$$

where $\boldsymbol{n}^b{}_{,0i} \equiv (\partial^2/\partial t \partial x_i)\boldsymbol{n}^b$, etc (Holz 1991). One easily obtains

$$\delta\Gamma_{\mathrm{CS}}(\lambda) = \lambda^2 \sum_{a=1}^{3} \sum_{i \leqslant j} s^a_i s^a_j \Phi(C^a_i, C^a_j) \tag{17}$$

where

$$\Phi(C^a_i, C^a_j) \equiv \frac{1}{4\pi} \int_{C^a_i} \int_{C^a_j} \frac{(\mathrm{d}\boldsymbol{x}^a_i \times \mathrm{d}\boldsymbol{x}^a_j) \cdot (\boldsymbol{x}^a_i - \boldsymbol{x}^a_j)}{|\boldsymbol{x}^a_i - \boldsymbol{x}^a_j|^3}$$

is the Gauss linking number and $\Phi \in \mathbb{Z}$; furthermore $s^a_i \in \mathbb{Z}$ and C^a_i represent strength and oriented loop of the ath singular disclination, respectively.

Use of the instanton number and its time derivative imply

$$\frac{\mathrm{d}\Gamma_{\mathrm{CS}}(\lambda)}{\mathrm{d}t} = 2\left(1 + \frac{2\lambda}{3}\right)\lambda^2 \sum_{a=1}^{3} \frac{2}{(8\pi)^2} \int_M \mathrm{d}^3x\, \boldsymbol{E}^a \cdot \boldsymbol{B}^a \tag{18a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\Gamma_{\mathrm{CS}}(\lambda) = 8\lambda^2 \sum_{a=1}^{3} \frac{2}{(8\pi)^2} \int_M \mathrm{d}^3x\, \delta\boldsymbol{E}^a \cdot \delta\boldsymbol{B}^a. \tag{18b}$$

Going over to the local rest frame one shows that $\delta \boldsymbol{E}^a \cdot \delta \boldsymbol{B}^a \neq 0$ only during processes, where singular disclinations intersect, and the same applies to $\boldsymbol{E}^a \cdot \boldsymbol{B}^a \neq 0$. Alternatively, it is easy to show using (13), that $\boldsymbol{E}^a \cdot \boldsymbol{B}^a \equiv 0$ holds for all smooth fields.

In the following section the topological properties of the $\mathrm{O}(3)/\mathbb{Z}_2$, and $\mathrm{SO}(3)/\mathrm{P}_i - \sigma$ models are worked out. In the nematic liquid ($\mathrm{O}(3)/\mathbb{Z}_2$-$\sigma$ model) S^2 is replaced by P^2. Hopf links may then be formed by disclinations of strengths $(n, m) \in \mathbb{Z}$ and $\mathbb{Z}+\frac{1}{2}$. A simple guess is that the Hopf invariant Q' of the nematic liquid is given by (20) with $(n, m) \in \frac{1}{2}\mathbb{Z}$ implying $|Q'| \geqslant \frac{1}{4}$ for non-trivial links.

Suppose a Hopf link is formed by two disclinations of strengths n, $m \in \mathbb{Z}+\frac{1}{2}$. Upon encircling each disclination twice each of the fields A_μ, $F_{\mu\nu}$ assumes its original value. We may therefore take four copies of the space M, provided with the cut surfaces Σ_n and Σ_m and glue them together along the oriented cut surfaces in such a manner that the A_μ, $F_{\mu\nu}$-fields change smoothly across the cut surfaces and are single valued in the space $M^* = \Pi_{i=1}^4 M^{(i)} \cup \Sigma_n^{(i)} \cup \Sigma_m^{(i)}$. This idea has been introduced by Dijkgraaf and Witten (1990) in a similar context. In M^* (branched over the Hopf-link in M) the strength of each disclination is doubled, i.e. $(n, m) \to (2n, 2m)$ and we obtain

$$Q^* = -\frac{1}{(8\pi)^2} \int_{M^*} \mathrm{d}^3x\, \boldsymbol{A} \cdot \boldsymbol{B} = 4nm\Phi(C_n, C_m).$$

with $Q^* \in \mathbb{Z}$. Alternatively one has

$$Q^* = -\sum_{i=1}^{4} \frac{1}{(8\pi)^2} \int_{M^{(i)}} \mathrm{d}^3x\, \boldsymbol{A} \cdot \boldsymbol{B} = 4Q' \tag{19}$$

because each term gives the same contribution due to $\boldsymbol{A} \cdot \boldsymbol{B} = (-\boldsymbol{A})(-\boldsymbol{B})$. Accordingly one obtains

$$Q' = \tfrac{1}{4}Q^* = nm\Phi(C_n, C_m) \tag{20}$$

which is the desired result, and extends additively in the presence of many links due to $\pi_3(P^2) \simeq Z$ being Abelian. Similarly for time-dependent processes one obtains

$$\frac{\mathrm{d}Q'}{\mathrm{d}t} = \frac{1}{2(8\pi)^2} \int_{M^*} \mathrm{d}^3x\, \boldsymbol{E} \cdot \boldsymbol{B}.$$

Consider next a biaxial liquid crystal described by a $\mathrm{SO}(3)/\mathrm{D}_2 - \sigma$ model, where D_2 is the dihedral group, consisting of the identity and the three identifications

$$\mathbf{R}(\boldsymbol{n}^1, \boldsymbol{n}^2, \boldsymbol{n}^3) \triangleq \mathbf{R}(\boldsymbol{n}^1, -\boldsymbol{n}^2, -\boldsymbol{n}^3) \triangleq \mathbf{R}(-\boldsymbol{n}^1, -\boldsymbol{n}^2, \boldsymbol{n}^3) \triangleq \mathbf{R}(-\boldsymbol{n}^1, \boldsymbol{n}^2, -\boldsymbol{n}^3) \tag{21}$$

which are compatible with the orientability of $\mathrm{SO}(3)/\mathrm{D}_2$. Due to $\mathrm{vol}(\mathrm{SO}(3)/\mathrm{D}_2) = \mathrm{vol}(\mathrm{SO}(3))/4$ it is more convenient to normalize Γ_{CS} with respect to the quotient space, i.e. we use

$$\Gamma_{\mathrm{CS}}^{\mathrm{D}_2}(\lambda) = 3\left(1 + \frac{2\lambda}{3}\right)\lambda^2 \mathcal{N}$$

where $\mathcal{N} \in \mathbb{Z}$ is the winding number related to $\pi_3(\mathrm{SO}(3)/\mathrm{D}_2)$. Similarly as described earlier we obtain alternatively

$$\Gamma_{\mathrm{CS}}^{\mathrm{D}_2}(\lambda) = 8\left(1 + \frac{2\lambda}{3}\right)\lambda^2 \sum_{a=1}^{3} Q_{\mathrm{D}_2}^a \tag{22}$$

yielding

$$Q_{\mathrm{D}_2}^a = -\frac{1}{(8\pi)^2} \int_M \mathrm{d}^3x\, \boldsymbol{A}_{\mathrm{D}_2}^{\prime a, \mathscr{H}} \cdot \boldsymbol{B}_{\mathrm{D}_2}^{\prime a, \mathscr{H}} = \mathcal{N}/8 \tag{23}$$

because each constituent of $\{\boldsymbol{n}^a\}_{a=1,2,3}$ gives the same contribution to (22). Accordingly each field is smoothed out in the product space M^* introduced above and complies with $Q^{*a} = 4Q^a_{\mathrm{D}_2} = \mathcal{N}/2$, where (15) has been used and being consistent with (23).

For a decomposition of the number $\mathcal{N}/8$ in (23) as a product of disclination strengths one writes (16) in the form

$$Q^a_{\mathrm{SO}(3)} = k \cdot h \qquad (k, h) \in \tfrac{1}{2}\mathbb{Z} \tag{24}$$

where $k \cdot h$ is the winding number with respect to S^3. With respect to that normalization one obtains in $\mathrm{SO}(3)/\mathrm{D}_2$-bundles disclinations of strength $p \in \frac{1}{4}\mathbb{Z}$, and $|Q^a_{\mathrm{SO}(3)/\mathrm{D}_2}| \geqslant \frac{1}{16}$ for non-trivial Hopf-links. More generally one obtains

$$Q^a_{\mathrm{SO}(3)/\mathrm{D}_2} = \sum_{i,j} p_i q_i \Phi(C_i, C_j) = \mathcal{N}/16 \qquad \mathcal{N} \in \mathbb{Z}, \quad (p_i, q_i) \in \tfrac{1}{4}\mathbb{Z} \tag{25a}$$

$$\Gamma^{\mathrm{SO}(3)/\mathrm{D}_2}_{\mathrm{CS}} = 2 * 8\left(1 + \frac{2\lambda}{3}\right)\lambda^2 \sum_{a=1}^{3} Q^a_{\mathrm{SO}(3)/\mathrm{D}_2} \tag{25b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma^{\mathrm{SO}(3)/\mathrm{D}_2}_{\mathrm{CS}} = 8\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\mathrm{CS}}(\lambda) \tag{25c}$$

where $\Gamma_{\mathrm{CS}}(\lambda)$ has been defined by (9). Observe that in M^* there are no singular disclinations, i.e. $\delta\Gamma^{\mathrm{SO}(3)/\mathrm{D}_2}_{\mathrm{CS}} \equiv 0$. This implies, that the topological theory in the multiply connected space M is of a qualitative different type as in the space M^*.

The present theory may also be extended to 3-spaces M with $\pi_1(M) \neq I$, as, for example, the 3-torus T^3. In that case the space (Witten 1989)

$$R(M) \to \mathrm{Hom}(\pi_1(M) \to \mathrm{SO}(3))/\mathrm{SO}(3)$$

modulo conjugation in SO(3), being isomorphic to the space of flat connections modulo gauge equivalence classes is non-trivial. However, the non-trivial flat connections will also have the representation (6a) for $\lambda = -1$, but differ in the glueings of the **R**-matrices yielding non-trivial holonomies $\mathbf{R}^t(2\pi)\mathbf{R}(0) \in \mathrm{SO}(3)$ along non-contractible loops (Holz 1991).

As an application of this formalism consider the computation of expection values of the holonomy operators $\mathrm{trace}_R\, P \exp\{-\oint_c \mathbf{A}_\mu\, \mathrm{d}x^\mu\}$ as a functional integral over equivalence classes of connections in the form

$$\langle W_R(\lambda)\rangle = \langle W_{R_1}(C_1) \ldots W_{R_n}(C_n)\rangle \tag{26}$$

with respect to Γ_{CS} (Witten 1989). In the following a modified version of (26) is presented for the $\mathrm{O}(3)/\mathbb{Z}_2 - \sigma$, and $\mathrm{SO}(3)/\mathrm{P}_i - \sigma$ models.

For the $\mathrm{O}(3)/\mathbb{Z}_2 - \sigma$ model functional integration in (26) will be defined with respect to M^* introduced above. In that space one may use

$$\langle W(L)\rangle_{\mathbb{Z}_2} = \int \mathrm{D}A \exp\left(\frac{\mathrm{i}2\pi k Q^*}{2^2}\right) \prod_{j=1}^{m} \exp\left(\mathrm{i} n_j \oint_{C_j} A_\mu\, \mathrm{d}x^\mu\right) \tag{27}$$

where k is a parameter and $\{n_j \in \mathbb{Z}\}_{j=1,\ldots,m}$ are representation numbers of U(1) (Witten 1989), *and* Q^* *is defined by (19). The factor* $1/2^2$ *in front of* Q^* *takes account of the* change-over from M to M^*. Functional integration in (27) leads to the same result as for the O(3)-σ model (Polyakov 1988), if k is replaced by $k/2^2$ in that formulae, i.e.

$$\langle W(L)\rangle_{\mathbb{Z}_2} = \exp\left\{\frac{\mathrm{i}8\pi}{k}\sum_{i,j} n_i n_j \Phi(C_i, C_j)\right\}. \tag{27$'$}$$

Due to $Q^* \in \mathbb{Z}$, k in (27) must be divisible by 4 in order that the measure in (27) is gauge invariant. Observe that for $k \in 4\mathbb{Z}$ the first exponential in (27) can be replaced by 1 and where according to (14), $A_\mu = -\boldsymbol{n}^1{}_{,\mu} \cdot \boldsymbol{n}^2$ can be used. Suitable boundary conditions at infinity guarantee $Q^* \in \mathbb{Z}$, and functional integration $\mathrm{D}A$ can be replaced by $\mathrm{D}\Omega^1\, \mathrm{D}\Omega^2$ over $S^2 \times S^2$, supplemented by the constraint $\boldsymbol{n}^1 \cdot \boldsymbol{n}^2 = 0$, and a gauge fixing, say $\nabla \cdot \boldsymbol{A} = 0$. Because now $\langle W(L') \rangle_{Z_2}$ is independent of k for $k \in 4\mathbb{Z}$, (27′) may be caused by $\boldsymbol{A}$-fields and their measure, used in functional integration, which do not correspond to the representation (14), necessary for the model to be interpreted as a σ-model. Because (27) is also defined for $k \in \mathbb{R}$, a possibility is that $4/k$ in (27′) has to be replaced by a periodic function in $k/4$, e.g. $\tan[\pi/4(1+k)]$. It can be expected that computation of $\langle W(L) \rangle_{Z_2}$ for the original space yields more interesting results.

For the SO(3)-σ model one may use a modified definition of (26)

$$\langle W(\lambda) \rangle_{\sigma,\lambda} \equiv \int \prod_{a=1}^{3} \mathrm{D}A^a(-1) \exp\left\{ \frac{\mathrm{i}2\pi k}{2^2} (3+2\lambda)\lambda^2 \sum_{a=1}^{3} \tfrac{1}{3} Q^{*a}_{\mathrm{SO}(3)} \right\}$$
$$\times f(\{A^a(-1)\}) \prod_{j=1}^{n} \operatorname{trace} P \exp\left\{ +\lambda \oint_{C_j} \mathbf{A}_\mu(-1)\, \mathrm{d}x^\mu \right\} \tag{28}$$

where the factor $1/2^2$ in front of $Q^{*a}_{\mathrm{SO}(3)}$ follows from (24). The functional $f(\{A^a(-1)\})$ represents the constraints (8). They may be taken account of by Lagrange multipliers and Fadeev–Popop ghosts. Note that together with (8)

$$Q^*_{\mathrm{SO}(3)} \equiv \tfrac{1}{3} \sum_{a=1} Q^{*a}_{\mathrm{SO}(3)} \in \mathbb{Z}.$$

Accordingly k must be divisible by 4, for the measure in (28) to be gauge invariant. This agrees with the result obtained by Dijkgraaf and Witten (1990).

Observe that (28) may also be written in the form

$$\langle W(\lambda) \rangle_{\sigma,\lambda} = \int \mathrm{D}S \exp\left(\frac{\mathrm{i}2\pi k}{2^2} (3+2\lambda)\lambda^2 Q^*_{\mathrm{SO}(3)} \right) \prod_{j=1}^{n} \operatorname{trace} P \exp\left\{ +\lambda \oint_{C_j} \mathbf{A}_\mu(-1)\, \mathrm{d}x^\mu \right\} \tag{28′}$$

where

$$A^a(-1) \equiv \hat{\Theta}^a\, \mathrm{d}\Theta + \sin\Theta\, \mathrm{d}\hat{\Theta}^a - (1-\cos\Theta)\varepsilon^{abc}\, \mathrm{d}\hat{\Theta}^b \hat{\Theta}^c$$

and $\mathrm{D}S$ represents functional integration over the 3-sphere, i.e., locally $\mathrm{D}S \triangleq 2(1-\cos\Theta)\, \mathrm{d}\Theta\, \mathrm{d}\Omega(\hat{\boldsymbol{\Theta}})$, $\Theta \in [0, 2\pi]$ and $\mathrm{d}\Omega(\hat{\boldsymbol{\Theta}})$ is the area element of the 2-sphere along the unit vector $\hat{\boldsymbol{\Theta}}$. Note that $\mathrm{vol}(S^3) = 16\pi^2$ in this normalization, and $Q^*_{\mathrm{SO}(3)} \in \mathbb{Z}$. The first exponential in (28′) assumes the value 1 for $\lambda = 0$ and the set $\Lambda = \{-\frac{3}{2}, \frac{1}{2}, \pm 1, 3\mathbb{Z}\}$ of λ-values. For all other λ-values one must use

$$Q^*_{\mathrm{SO}(3)} = -\frac{1}{16\pi^2} \int_{M_3^*} A^1(-1) A^2(-1) A^3(-1) \in \mathbb{Z}.$$

Observe, however, that computation of (28′) for $\lambda \in \mathbb{R}$ and analytic continuation to $\lambda \in \Lambda$ may lead to a different result from that obtained via the method explained above (see similar comment below (27)). For $\lambda = -1$ the holonomy operator is trivial, i.e. $P \exp(-\int_1^2 A(-1)) = R^t(2) \cdot R(1)$ and therefore $\langle W(L) \rangle_{\sigma,-1} = \langle W(L) \rangle_{\sigma,0}$.

As a normalization of (28′) one may set $\langle W(L) \rangle_{\sigma,0} = 1$. Due to

$$\langle W(L) \rangle_{\sigma,\lambda} = \langle W(-L) \rangle_{\sigma,-\lambda} \qquad \lambda = 0, 1, 3\mathbb{Z} \quad \text{and} \quad \lambda = \tfrac{1}{2} \quad \text{for } k \in 8\mathbb{Z} \tag{29}$$

where $L \to -L$ implies $\{C_j \to -C_j\}_{j=1,\ldots,n}$, and $\lambda \neq 0, -1$ are non-flat connections. The role played by λ in (28′) is remotely reminiscent of Θ-statistics in anyon physics.

In the computation of $\langle W(L)\rangle_{\sigma,\lambda}$ for $G \simeq SO(3)/D_2$ the factor $1/2^2$ in (28) and (28′) is replaced by $1/4^2$ as follows from (25*a*), and k must be divisible by 4^2. For $\mathrm{ord}(P_i) > 4$ the situation is slightly more involved. For the octahedral group O of $\mathrm{ord}(O) = 24$, there exist two-, three- and four-fold axes of symmetry. Here one has to replace in (25) $D_2 \to O$, use in (25*a*)

$$(p_i, q_i) \in \left(\frac{1}{2*2}\mathbb{Z}, \frac{1}{2*3}\mathbb{Z}, \frac{1}{2*4}\mathbb{Z}\right)$$

in such a fashion that $Q^a_{SO(3)/O} = \mathcal{N}/96$, $\mathcal{N} \in \mathbb{Z}$, and replace the factor 8 in (25*b*) by $\mathrm{ord}(\mathbb{Z}_2 \times O) = 48$. This implies that in (28) one has to replace $Q^*_{SO(3)}$ by $Q^{*a}_{SO(3)/O}$ and in (28′) $Q^*_{SO(3)}$ by

$$Q^*_{SO(3)/O} = \tfrac{1}{3}\sum_{a=1}^{3} Q^{*a}_{SO(3)/O}.$$

In addition the factor $1/2^2$ in (28) and (28′) has to be replaced by 1/96. This extends to the icosahedral group I of $\mathrm{ord}(I) = 60$, where the characteristic number 48 of O is replaced by 120. Due to the presence of a 5-fold axis of symmetry in I, we have to use in (25*a*),

$$(p_i, q_i) \in \left(\frac{1}{2*2}\mathbb{Z}, \frac{1}{2*3}\mathbb{Z}, \frac{1}{2*4}\mathbb{Z}, \frac{1}{2*5}\mathbb{Z}\right)$$

in such a fashion that $Q^a_{SO(3)/I} = \mathcal{N}/120$ results, $\mathcal{N} \in \mathbb{Z}$.

Note that the present theory may be extended to the groups $G = SO(N)/P_i(N)$, where $N \geq 4$, $P_i(N) \subset SO(N)$, using $\mathbf{R}(\{\boldsymbol{n}^a\}_{a=1,\ldots,N}) \in SO(N)$ and $\{\boldsymbol{n}^a\}_{a=1,\ldots,N}$ is an orthonormal N-tuple, and $\pi_3(SO(N)) = \mathbb{Z}+\mathbb{Z}$, $\mathbb{Z}$ for $N = 4$ and $N \geq 5$, respectively (Whitehead 1978). Accordingly for $P_i = I$, $\langle W(L)\rangle_{\sigma,\lambda}$ depends on three parameters k, N and λ, and in a more general approach λ may be replaced by $\{\lambda^a(x)\}$ as indicated below (8). Accordingly one may conjecture that

$$\int D\Lambda \langle W(L)\rangle_{\sigma,\{\lambda^a\}} \sim \langle W(L)\rangle \tag{30}$$

holds, where $D\Lambda$ is a functional integration over λ-connections. Observe that integration over DS and $D\Lambda$ in (30) implies six degrees of freedom per space point, which is the same number as used in (26) due to $A_0 \equiv 0$ (Witten 1989).

In conclusion the close topological relation between the O(3)-σ and SO(3)-σ models derived may prove useful for the study of a semi-classical approximation to the Heisenberg antiferromagnets, which for spin $s \to \infty$ and $s \to \frac{1}{2}$ may be described by the O(3)-σ and SO(3)-σ models as speculated recently (Holz 1990). Evaluation of expectation values of holonomy operators of the σ-models discussed here is faced with certain shortcomings, because topology changing processes imply the formation of singularities due to $\pi_2(S^2)$, $\pi_2(P^2) \simeq \mathbb{Z}$, leading to non-trivial U(1)-connections of the drei-bein's constituent fields. An extension of the theory taking account of 'magnetic' N-pole singularities is of interest in connection with high-T_c superconducitvity (Holz and Gong 1988) and is in progress.

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